

SUBGROUP CORRESPONDENCES

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Dedicated to the memory of Ola Bratteli

ABSTRACT. For a closed subgroup of a locally compact group the Rieffel induction process gives rise to a C^* -correspondence over the C^* -algebra of the subgroup. We study the associated Cuntz-Pimsner algebra and show that, by varying the subgroup to be open, compact, or discrete, there are connections with the Exel-Pardo correspondence arising from a cocycle, and also with graph algebras.

1. INTRODUCTION

Let H be a closed subgroup of a locally compact group G . Then the Rieffel induction process involves a $C^*(G) - C^*(H)$ correspondence X , and restricting to H (more precisely, composing the left $C^*(G)$ -module structure on X with the canonical homomorphism from $C^*(H)$ into $M(C^*(G))$) makes X into a correspondence over $C^*(H)$. We examine properties of the Cuntz-Pimsner algebra of this correspondence in terms of how H sits as a subgroup of G .

The $C^*(H)$ -correspondence X has some special properties, e.g., it is nondegenerate and full. Our results are sharpest when X is *regular*, i.e., $C^*(H)$ acts on the left faithfully by compact operators, which seems to entail H being open and of finite index in G . In this case the representations of the Cuntz-Pimsner algebra \mathcal{O}_X correspond to representations U of H together with an explicit unitary equivalence between U and $(\text{Ind}_H^G U)|_H$. If H is open and central in G , then the Cuntz-Pimsner algebra \mathcal{O}_X is the tensor product of $C^*(H)$ and a Cuntz algebra.

When G is discrete, any choice of cross section of G/H in G gives rise to a cocycle for the action of H on G/H by translation, and \mathcal{O}_X is isomorphic to an associated Exel-Pardo algebra (for the action of H on a directed graph with one vertex), generated by a Cuntz algebra

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and a representation of H whose interaction with the Cuntz algebra involves the cocycle. Alternatively, this is described by a self-similar action of H in the sense of Nekrashevych. The cohomology class of the cocycle seems to be determined by the subgroup H itself, explaining the independence of \mathcal{O}_X upon the choice of cross section.

When the subgroup H is compact the Peter-Weyl theorem says that $C^*(H)$ is a c_0 -direct sum of finite-dimensional algebras, so is Morita equivalent to a commutative C^* -algebra with the same spectrum as H . It follows that, by a theorem of Muhly and Solel, \mathcal{O}_X is Morita equivalent to the Cuntz-Pimsner algebra of a correspondence over this commutative algebra, and hence (by a result of Patani and the first and third authors) to the C^* -algebra of a directed graph E that can be computed in terms of multiplicities of irreducible representations of H induced across the correspondence X . If H is already abelian, then \mathcal{O}_X is isomorphic to this graph algebra $C^*(E)$.

In Section 7 we specialize further to a finite group G . Then Mackey's Subgroup Theorem allows us to compute the multiplicities (and hence the directed graph E) using the double H -cosets. It turns out that interesting examples arise even when H has order 2, and we examine this case in some detail. $C^*(E)$, and hence \mathcal{O}_X , is a UCT Kirchberg algebra that is classifiable by its K -theory, which only depends upon how large the centralizer of H is in G , more precisely upon two positive integers r and q , where the first is the index of H in its centralizer and $r + 2q$ is the index of H in G . When $r = 1$ we have $K_0 = \mathbb{Z}_q \oplus \mathbb{Z}$ and $K_1 = \mathbb{Z}$, and it follows (taking into account also the class of the identity in K_0) that \mathcal{O}_X is isomorphic to the C^* -algebra of the category of paths given by the positive submonoid of a Baumslag-Solitar group, studied by Spielberg. When q is also 1, \mathcal{O}_X is Morita equivalent to two C^* -algebras studied by Laca and Spielberg, involving a projective linear group acting on the boundary of the upper half plane or alternatively the Ruelle algebra of a 2-adic solenoid. On the other hand, when $r > 1$ the K_1 group is trivial, and the K_0 group depends upon whether $r + q - 1$ and q are coprime. If they are coprime, the K_0 group is finite cyclic, and hence \mathcal{O}_X is a matrix algebra over a Cuntz algebra. But if $r + q - 1$ and q are not coprime then K_0 is a direct sum of two finite cyclic groups, and unfortunately we do not know any other famous Kirchberg algebras with this K -theory.

In the last section we briefly discuss a curious connection with Doplicher-Roberts algebras studied by Mann, Raeburn, and Sutherland. The situation is decidedly different (in particular, not involving induced representations), but the results are uncannily similar.

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2. PRELIMINARIES

We record our notation and conventions for C^* -correspondences. First of all, if X is an A -correspondence, with left A -module structure given by the homomorphism $\varphi = \varphi_A: A \rightarrow \mathcal{L}(X)$, we will freely switch back and forth between the notations ax and $\varphi(a)x$ for $a \in A, x \in X$. We call the correspondence X *faithful* if φ is faithful, and *nondegenerate* if $\varphi(A)X = X$.

A (Toeplitz) representation of X in a C^* -algebra B is a pair (ψ, π) , where $\psi: X \rightarrow B$ is a linear map and $\pi: A \rightarrow B$ is a homomorphism such that for $a \in A, x, y \in X$ we have

$$\begin{aligned}\psi(ax) &= \pi(a)\psi(x) \\ \psi(x)^*\psi(y) &= \pi(\langle x, y \rangle_A)\end{aligned}$$

(and hence $\psi(xa) = \psi(x)\pi(a)$). If \mathcal{H} is a Hilbert space and B is the algebra $B(\mathcal{H})$ of bounded operators on \mathcal{H} , we say (ψ, π) is a representation of X on \mathcal{H} . A representation (ψ, π) of X on a Hilbert space \mathcal{H} is *nondegenerate* if the C^* -algebra generated by $\psi(X) \cup \pi(A)$ acts nondegenerately on \mathcal{H} . If X is nondegenerate as a correspondence then a representation (ψ, π) of X on \mathcal{H} is nondegenerate if and only if the representation π of A is nondegenerate.

The *Toeplitz algebra* \mathcal{T}_X of X is universal for Toeplitz representations. $\mathcal{K}(X)$ denotes the algebra of (generalized) compact operators on X , which is the closed linear span of the (generalized) rank-one operators $\theta_{x,y}$ given by $\theta_{x,y}z = x\langle y, z \rangle_A$. For any representation (ψ, π) of X in B , there is a unique homomorphism $\psi^{(1)}: \mathcal{K}(X) \rightarrow B$ such that $\psi^{(1)}(\theta_{x,y}) = \psi(x)\psi(y)^*$ for all $x, y \in X$.

The *Katsura ideal* of A is $J_X := \varphi^{-1}(\mathcal{K}(X)) \cap (\ker \varphi)^\perp$, where for any ideal I of A the *orthogonal complement* of I is the ideal $I^\perp := \{a \in A : a = 0 \text{ for all } b \in I\}$. A representation (ψ, π) of X in B is *Cuntz-Pimsner covariant* if $\pi(a) = \psi^{(1)} \circ \varphi(a)$ for all $a \in J_X$, and the *Cuntz-Pimsner algebra* \mathcal{O}_X is universal for Cuntz-Pimsner covariant representations, and is generated as a C^* -algebra by a universal Cuntz-Pimsner covariant representation (k_X, k_A) . For any Cuntz-Pimsner covariant representation (ψ, π) of X , we write $\psi \times \pi$ for the unique

homomorphism of \mathcal{O}_X satisfying

$$\psi = (\psi \times \pi) \circ k_X \quad \text{and} \quad \pi = (\psi \times \pi) \circ k_A.$$

If X is nondegenerate as an A -correspondence, then the homomorphism $k_A: A \rightarrow \mathcal{O}_X$ is nondegenerate in the sense that $k_A(A)\mathcal{O}_X = \mathcal{O}_X$.

Our primary object of study will be the Cuntz-Pimsner algebra of a correspondence over the C^* -algebra of a subgroup H of a locally compact group G . Thus it is relevant to consider what sorts of representations of H will occur as part of a Cuntz-Pimsner covariant representation. The remainder of this section is devoted to some general remarks concerning representations of C^* -correspondences. We claim no originality for these — they are either readily available in the literature, or folklore.

Lemma 2.1. *The Toeplitz representations of an A -correspondence X on a Hilbert space \mathcal{H} are in 1-1 correspondence with the pairs (π, V) , where π is a representation of A on \mathcal{H} and $V: X \otimes_A \mathcal{H} \rightarrow \mathcal{H}$ is an isometry implementing a unitary equivalence between $X\text{-Ind } \pi$ and a subrepresentation of π .*

Proof. First let (ψ, π) be a representation of X on \mathcal{H} . The Rieffel induction process yields a representation of $\mathcal{L}(X)$ on $B(X \otimes_A \mathcal{H})$, and composing with the left-module homomorphism $\varphi: A \rightarrow \mathcal{L}(X)$ gives an induced representation $X\text{-Ind } \pi: A \rightarrow \mathcal{L}(X \otimes_A \mathcal{H})$.

Borrowing from [FR99], we can define an isometry $V: X \otimes_A \mathcal{H} \rightarrow \mathcal{H}$ by

$$V(x \otimes \xi) = \psi(x)\xi \quad \text{for } x \in X, \xi \in \mathcal{H}.$$

Conjugating by V , [FR99, Proposition 1.6] gives a unique representation $\rho: \mathcal{L}(X) \rightarrow B(\mathcal{H})$ with essential subspace

$$\mathcal{H}_\psi := \overline{\text{span}}\{\psi(X)\mathcal{H}\} = \text{ran } V$$

such that

$$\rho(T)\psi(x)\xi = \psi(Tx)\xi \quad \text{for } T \in \mathcal{L}(X), x \in X, \xi \in \mathcal{H},$$

and moreover $\rho(\theta_{x,y}) = \psi(x)\psi(y)^*$.

A quick computation shows that the diagram

$$\begin{array}{ccc} X \otimes_A \mathcal{H} & \xrightarrow{X\text{-Ind } \pi(a)} & X \otimes_A \mathcal{H} \\ \downarrow V & & \downarrow V \\ \mathcal{H} & \xrightarrow{\pi(a)} & \mathcal{H} \end{array}$$

commutes. This also shows that (ψ, π) is (Cuntz-Pimsner) covariant on the invariant subspace \mathcal{H}_ψ , because if $\varphi(a) \in \mathcal{K}(X)$ then for all $x \in X$ we have

$$\psi^{(1)}(\varphi(a))\psi(x) = \rho(\varphi(a))\psi(x) = \psi(ax) = \pi(a)\psi(x).$$

Thus V implements a unitary equivalence between X -Ind π and a subrepresentation of π , namely the restriction of π to \mathcal{H}_ψ .

Conversely, suppose we have a representation $\pi: A \rightarrow B(\mathcal{H})$ an isometry $V: X \otimes_A \mathcal{H} \rightarrow \mathcal{H}$, with range L , such that

$$\text{Ad } V \circ X\text{-Ind } \pi(a)\xi = \pi(a)\xi \quad \text{for all } a \in A, \xi \in L.$$

We will show that π is part of a representation (ψ, π) of X on \mathcal{H} such that $V(x \otimes \xi) = \psi(x)\xi$ for $x \in X, \xi \in \mathcal{H}$. For $x \in X$, define $\psi(x): \mathcal{H} \rightarrow \mathcal{H}$ by

$$\psi(x)\xi = V(x \otimes \xi).$$

Then $\psi(x)$ is a linear operator, and is bounded because

$$\begin{aligned} \|\psi(x)\xi\| &= \|V(x \otimes \xi)\| \\ &= \|x \otimes \xi\| \quad (\text{since } V \text{ is isometric}) \\ &= \|x\|\|\xi\|. \end{aligned}$$

Moreover, it is obvious that $\psi: X \rightarrow B(\mathcal{H})$ is linear.

Here are the routine computations verifying that the pair (ψ, π) is a representation of X : if $a \in A$ and $x \in X$, then for all $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \psi(ax)\xi &= V(ax \otimes \xi) \\ &= VX\text{-Ind } \pi(a)(x \otimes \xi) \\ &= \pi(a)V(x \otimes \xi) \\ &= \pi(a)\psi(x)\xi, \end{aligned}$$

so $\psi(ax) = \pi(a)\psi(x)$, and if $x, y \in X$ then for all $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned} \langle \psi(x)^*\psi(y)\xi, \eta \rangle &= \langle \psi(y)\xi, \psi(x)\eta \rangle \\ &= \langle V(y \otimes \xi), V(x \otimes \eta) \rangle \\ &= \langle y \otimes \xi, x \otimes \eta \rangle \quad (\text{since } V \text{ is isometric}) \\ &= \langle \xi, \pi(\langle y, x \rangle_A)\eta \rangle \\ &= \langle \pi(\langle x, y \rangle_A)\xi, \eta \rangle, \end{aligned}$$

so $\psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A)$.

Finally, it's obvious from the constructions that if we now start with this newly manufactured ψ then the intertwining isometry defined as in the first part of the proof agrees with V . \square

Before considering the Cuntz-Pimsner covariant representations, we specialize the correspondence:

Definition 2.2. We call an A -correspondence X *regular* if $J_X = A$, i.e., A acts faithfully by compact operators on X .

Remark 2.3. If X is nondegenerate and regular, then $k_X^{(1)}: \mathcal{K}(X) \rightarrow \mathcal{O}_X$ is nondegenerate, because k_A is.

Recall that for a representation (ψ, π) of X on a Hilbert space \mathcal{H} we write $\mathcal{H}_\psi = \overline{\text{span}}\{\pi(X)\mathcal{H}\}$.

Lemma 2.4. *A nondegenerate representation (ψ, π) of a nondegenerate regular A -correspondence X on a Hilbert space \mathcal{H} is Cuntz-Pimsner covariant if and only if $\mathcal{H}_\psi = \mathcal{H}$.*

Proof. First assume that $\mathcal{H}_\psi = \mathcal{H}$. Then [FMR03, Remark 6.5] says that (ψ, π) is Cuntz-Pimsner covariant. However, there is a subtlety: in [FMR03] the definition of Cuntz-Pimsner covariance only involves $\varphi^{-1}(\mathcal{K}(X))$. We can sort this out by noting that [FMR03, Remark 6.5] refers back to [FMR03, Lemma 1.9], which implies that (ψ, π) is coisometric on the ideal J_X of $\varphi^{-1}(\mathcal{K}(X))$ if and only if

$$(2.1) \quad \pi(J_X)\mathcal{H} \subset \mathcal{H}_\psi.$$

Since $\mathcal{H}_\psi = \mathcal{H}$, the condition (2.1) is automatically satisfied, so (ψ, π) is coisometric on J_X , i.e., is covariant in the modern (Katsura) sense. Actually, since X is regular by hypothesis, we have $J_X = \varphi^{-1}(\mathcal{K}(X))$, so [FMR03, Remark 6.5] applies without any modification.

Conversely, assume that (ψ, π) is Cuntz-Pimsner covariant. Since π is nondegenerate, it suffices to show that

$$\pi(A) \subset \overline{\text{span}}\{\psi(X)\psi(X)^*\}.$$

But since $J_X = A$, by Cuntz-Pimsner covariance we have

$$\begin{aligned} \pi(A) &\subset \psi^{(1)}(\mathcal{K}(X)) = \overline{\text{span}}\{\psi^{(1)}(\theta_{x,y}) : x, y \in X\} \\ &= \overline{\text{span}}\{\psi(X)\psi(X)^*\}. \end{aligned} \quad \square$$

Corollary 2.5. *The Cuntz-Pimsner covariant representations of a nondegenerate regular A -correspondence X on a Hilbert space \mathcal{H} are in 1-1 correspondence with the pairs (π, V) , where π is a representation of A on \mathcal{H} and $V: X \otimes_A \mathcal{H} \rightarrow \mathcal{H}$ implements a unitary equivalence between $X\text{-Ind } \pi$ and π .*

3. SUBGROUPS

Now let H be a closed subgroup of a locally compact group G , and let X be the $C^*(G)$ – $C^*(H)$ correspondence for Rieffel induction. Composing the left $C^*(G)$ -module structure with the canonical nondegenerate homomorphism $C^*(H) \rightarrow M(C^*(G))$, X becomes a C^* -correspondence over $A := C^*(H)$.

Note that the left-module homomorphism $\varphi = \varphi_A: A \rightarrow \mathcal{L}(X)$ is nondegenerate, and the A -correspondence X is full. It still seems to be unknown (at least to us) whether the correspondence X is always *faithful* in the sense that φ_A is faithful, equivalently whether the canonical homomorphism $C^*(H) \rightarrow M(C^*(G))$ is faithful (see [Rie74, paragraph following Proposition 4.1]. It is faithful if the subgroup H is either open [Rie74, Proposition 1.2] or compact (this follows from [Fel64, Corollary 3 of Theorem 5.5]), and also if H is amenable, since then $C^*(H) = C_r^*(H)$ and the composition

$$C_r^*(H) \rightarrow M(C^*(G)) \rightarrow M(C_r^*(G))$$

is always faithful. It seems to us that examples where $C^*(H) \rightarrow M(C^*(G))$ is not faithful, if they exist, will be somewhat exotic.

Hypothesis 3.1. We will tacitly assume throughout that the subgroup H of G is such that the correspondence X is faithful, equivalently $C^*(H) \rightarrow M(C^*(G))$ is faithful.

Question 3.2. When will φ_A map $C^*(H)$ into the algebra $\mathcal{K}(X)$ of compact operators?

Note that the imprimitivity theorem says

$$\mathcal{K}(X) = C_0(G/H) \rtimes G.$$

If H is open then the natural inclusion $C_c(H) \hookrightarrow C_c(G)$ extends to a faithful embedding $C^*(H) \subset C^*(G)$ [Rie74, Proposition 1.2]. If H is cocompact in G , i.e., G/H is compact, then $C_0(G/H) = C(G/H)$ is unital, so $i_G(C^*(G)) \subset C(G/H) \rtimes G$. So, if H is open and cocompact then $\varphi(A) \subset \mathcal{K}(X)$.

On the other hand, if H is not cocompact, then $C_0(G/H)$ is not unital, and it follows from Lemma 3.3 below that $\varphi(A) \cap \mathcal{K}(X) = \{0\}$. If H is cocompact but not open, the situation is not clear to us in general, and we will not seriously study this case.

In the preceding paragraph we appealed to the following lemma, which must be folklore:

Lemma 3.3. *If α is an action of a locally compact group G on a nonunital C^* -algebra A , then*

$$i_G(M(C^*(G))) \cap (A \rtimes_\alpha G) = \{0\}.$$

Proof. First note that it suffices to show that $i_G(C^*(G)) \cap (A \rtimes G) = \{0\}$, because then if we had any nonzero $m \in M(C^*(G))$ for which $i_G(m) \in A \rtimes G$, then we could choose $c \in C^*(G)$ such that $mc \neq 0$, and then $i_G(mc)$ would be a nonzero element of $i_G(C^*(G)) \cap (A \rtimes G)$.

The action extends continuously to the unitization \tilde{A} , and we have a split short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \begin{array}{c} \xleftarrow{\sigma} \\ \xrightarrow{\rho} \end{array} \mathbb{C} \longrightarrow 0$$

that is G -equivariant. Taking crossed products, we get a split short exact sequence

$$0 \longrightarrow A \rtimes G \xrightarrow{\iota \rtimes G} \tilde{A} \rtimes G \begin{array}{c} \xleftarrow{i_G^{\tilde{A}}} \\ \xrightarrow{\rho \rtimes G} \end{array} C^*(G) \longrightarrow 0,$$

where $i_G^{\tilde{A}}: G \rightarrow \tilde{A} \rtimes G$ is the canonical homomorphism, which coincides with $\sigma \rtimes G$.

The canonical covariant pair $(i_A, i_G^A): (A, G) \rightarrow M(A \rtimes_\alpha G)$ is compatible with the pair $(i_{\tilde{A}}, i_G^{\tilde{A}})$ in the following sense: first of all, the nondegenerate homomorphism $i_A: A \rightarrow M(A \rtimes G)$ extends canonically to $\tilde{i}_A: \tilde{A} \rightarrow M(A \rtimes G)$, the pair $(\tilde{i}_A, i_G^A): (\tilde{A}, G) \rightarrow M(A \rtimes G)$ is covariant and the diagram

$$\begin{array}{ccc} A \rtimes G & \xrightarrow{\iota \rtimes G} & \tilde{A} \rtimes G \\ \downarrow & \tilde{i}_A \times i_G^A \swarrow & \uparrow i_G^{\tilde{A}} \\ M(A \rtimes G) & \xleftarrow{i_G^A} & C^*(G) \end{array}$$

commutes. Combining diagrams, if we had a nonzero $c \in C^*(G)$ such that $i_G^A(c) \in A \rtimes G$, then $i_G^{\tilde{A}}(c)$ would be a nonzero element of $\tilde{A} \rtimes G$ that lies in the ideal $A \rtimes G$, which would give

$$0 \neq c = (\rho \rtimes G) \circ i_G^{\tilde{A}}(c) = (\rho \rtimes G) \circ (\iota \rtimes G) \circ i_G^A(c) = 0,$$

a contradiction. \square

Corollary 3.4. *When H is open, we have a dichotomy: G/H is either finite, in which $J_X = A$, or infinite, in which case $J_X = \{0\}$ and $\mathcal{O}_X = \mathcal{T}_X$.*

Remark 3.5. In any case, if H is cocompact in G and (ψ, π) is a Toeplitz representation of the A -correspondence X on a Hilbert space \mathcal{H} , then for $a \in A, x \in X$ we have

$$\pi(a)\psi(x) = \psi(ax) = \psi^{(1)}(\varphi(a))\psi(x),$$

so the restriction of (ψ, π) to the invariant subspace \mathcal{H}_ψ is Cuntz-Pimsner covariant.

Here are the two extremes for how H can sit inside G : if $H = \{1\}$, then X is the Hilbert space $L^2(G)$, regarded as a \mathbb{C} -correspondence, so \mathcal{O}_X is the Cuntz algebra $\mathcal{O}_{L^2(G)}$. Note that, due to our standing hypothesis that G is second countable, the Hilbert space $L^2(G)$ is separable, and so $\mathcal{O}_{L^2(G)}$ is either the Cuntz algebra \mathcal{O}_n if G is finite of order n , or \mathcal{O}_∞ if G is infinite. At the other extreme, if $H = G$, then X is the *identity* $C^*(G)$ -correspondence $C^*(G)$, so $\mathcal{O}_X = C(\mathbb{T}) \otimes C^*(G)$.

Here are a couple of obvious general properties of X and \mathcal{O}_X . If H is exact, then so is $C^*(H)$, so \mathcal{O}_X is exact by [Kat04, Theorem 7.1]. Similarly, if H is amenable, or more generally if $C^*(H)$ is nuclear, then \mathcal{O}_X is nuclear, by [Kat04, Corollary 7.4].

Remark 3.6. Since we are assuming that $\varphi_A: A \rightarrow \mathcal{L}(X)$ is nondegenerate, for a representation (ψ, π) of X on a Hilbert space \mathcal{H} to be nondegenerate is equivalent to nondegeneracy of π . Motivated by this, when we refer to a Toeplitz representation (ψ, π) of X on a Hilbert space, we tacitly assume that π , and hence (ψ, π) , is nondegenerate.

4. H OPEN

Suppose that H is an open subgroup of G . Then every double H -coset HtH is open, and $C_c(HtH)$ is closed under left and right multiplication by $C_c(H)$. We have

$$C_c(G) = \bigodot_{HtH \in H \backslash G / H} C_c(HtH),$$

where \odot refers to the algebraic direct sum, i.e., with finitely nonzero coordinates. If $f \in C_c(HtH)$ and $g \in C_c(HtH)$ then

$$\begin{aligned} \langle f, g \rangle_A(h) &= (f^* * g)(h) \\ &= \int_G f^*(r)g(r^{-1}h) dr \\ &= \int_G \overline{f(r^{-1})}\Delta(r^{-1})g(r^{-1}h) dr \\ &= \int_G \overline{f(r)}g(rh) dr \end{aligned}$$

$$= \int_{HtH} \overline{f(r)} g(rh) dr$$

which is 0 unless $HtH = HtH$. It follows that the norm closures X_{HtH} of the sets $C_c(HtH)$ in X are mutually orthogonal A -subcorrespondences, and we get a decomposition

$$X = \bigoplus_{HtH \in H \backslash G/H} X_{HtH}$$

of correspondences. This might be of use in later investigations, but at present we only apply it to the following special case.

Proposition 4.1. *Let H be open and normal in G . Choose a cross section $\eta: G/H \rightarrow G$, with $\eta(H) = 1$. Let $A = C^*(H)$. For each $tH \in G/H$, let A_{tH} be the A -correspondence associated to the automorphism $\text{Ad } \eta(tH)$ of A , i.e., it is the standard Hilbert A -module A but with left A -module structure given by*

$$a \cdot_{tH} b = \text{Ad } \eta(tH)(a)b \quad \text{for } a, b \in A.$$

Then

$$X \simeq \bigoplus_{tH \in G/H} A_{tH}.$$

as A -correspondences.

Proof. Since H is normal, the double cosets HtH are just cosets tH , so by the discussion preceding the proposition We have a decomposition

$$X = \bigoplus_{tH \in G/H} X_{tH},$$

where X_{tH} is the closure of $C_c(tH)$ in X . It now suffices to show that for all $tH \in G/H$ we have $X_{tH} \simeq A_{tH}$ as A -correspondences.

We use the conventions from [RW98, Theorem C.23] for the correspondence X ; more precisely, the formulas in [RW98] are for the $(C_0(G/H) \rtimes G) - C^*(H)$ imprimitivity bimodule structure on X , and we can restrict the left module multiplication to $C^*(G)$, and then we restrict further to the subalgebra $A = C^*(H)$. Define a linear map $\Psi: C_c(tH) \rightarrow C_c(H)$ by

$$(\Psi x)(h) = x(\eta(tH)h).$$

Then for $k, h \in H$ and $x \in C_c(tH)(G)$ we have

$$\begin{aligned} (\Psi kx)(h) &= (kx)(\eta(tH)h) \\ &= x(k^{-1}\eta(tH)h) \\ &= x(\eta(tH)\eta(tH)^{-1}k^{-1}\eta(tH)h) \end{aligned}$$

$$\begin{aligned}
&= (\Psi x)(\text{Ad } \eta(tH)^{-1}(k)^{-1}h) \\
&= (\text{Ad } \eta(tH)^{-1}(k)\Psi x)(h) \\
&= (k \cdot_{tH} \Psi x)(h)
\end{aligned}$$

and

$$\begin{aligned}
(\Psi xk)(h) &= (xk)(\eta(tH)h) \\
&= x(\eta(tH)hk^{-1})\Delta(k^{-1}) \\
&= (\Psi x)(hk^{-1})\Delta(k^{-1}) \\
&= ((\Psi x)k)(h),
\end{aligned}$$

from which it follows that Ψ preserves the $C_c(H)$ -bimodule structure.

For the inner products, if $x, y \in C_c(tH)(G)$ and $h \in H$ then

$$\begin{aligned}
\langle \Psi x, \Psi y \rangle_A(h) &= ((\Psi x)^* * \Psi y)(h) \\
&= \int_H (\Psi x)^*(k)(\Psi y)(k^{-1}h) dk \\
&= \int_H \overline{(\Psi x)(k^{-1})} \Delta(k^{-1}) y(\eta(tH)k^{-1}h) dk \\
&= \int_H \overline{x(\eta(tH)k^{-1})} \Delta(k^{-1}) y(\eta(tH)k^{-1}h) dk \\
&= \int_H \overline{x(\eta(tH)k)} y(\eta(tH)kh) dk \\
&= \int_{tH} \overline{x(t)} y(th) dt \quad (\text{since } \eta(tH)H = tH) \\
&= \int_G \overline{x(t)} y(th) dt \quad (\text{since } \text{supp } x \cup \text{supp } y \subset tH) \\
&= \int_G \overline{x(t^{-1})} \Delta(t^{-1}) y(t^{-1}h) dt \\
&= \int_G x^*(t) y(t^{-1}h) dt \\
&= (x^* * y)(h) \\
&= \langle x, y \rangle_A(h).
\end{aligned}$$

Thus Ψ is an isomorphism of A -correspondences, and we are done. \square

And now we specialize even further:

Corollary 4.2. *If H is open and central in G , then*

$$X \simeq \ell^2(G/H) \otimes A,$$

where on the right-hand side we mean the external tensor product of the \mathbb{C} -correspondence $\ell^2(G/H)$ and the standard A -correspondence A . Consequently, by Lemma 4.3 below,

$$\mathcal{O}_X \simeq \mathcal{O}_{\ell^2(G/H)} \otimes A.$$

Corollary 4.2 referred to the following lemma, which is probably folklore, although we could not find a convenient reference for it. It could almost (but not quite) be deduced from [Mor, Theorem 5.4], but our special case is much more elementary.

Lemma 4.3. *Let A be a C^* -algebra, let \mathcal{H} be a Hilbert space, and let $\mathcal{H} \otimes A$ be the A -correspondence given by the external tensor product of the \mathbb{C} -correspondence \mathcal{H} and the standard A -correspondence A . Then*

$$\mathcal{O}_{\mathcal{H} \otimes A} \simeq \mathcal{O}_{\mathcal{H}} \otimes A.$$

Proof. Let $k = k_{\mathcal{H}}: \mathcal{H} \rightarrow \mathcal{O}_{\mathcal{H}}$ be the universal map. Consider the linear map $k \otimes \text{id}: \mathcal{H} \odot A \rightarrow \mathcal{O}_{\mathcal{H}} \otimes A$, where $\text{id} = \text{id}_A$. Elementary computations show that the pair $(k \otimes \text{id}, 1 \otimes \text{id})$, where 1 is the identity map on \mathcal{H} , is a representation of the pre-correspondence $\mathcal{H} \odot A$ in $\mathcal{O}_{\mathcal{H}} \otimes A$. It follows that $k \otimes \text{id}$ extends uniquely to a bounded linear map from $\mathcal{H} \otimes A$ to $\mathcal{O}_{\mathcal{H}} \otimes A$.

Case 1. $\dim \mathcal{H} = \infty$. Then $J_{\mathcal{H}} = \{0\}$. It follows that $J_{\mathcal{H} \otimes A} = \{0\}$, since the left module map $\varphi: A \rightarrow \mathcal{L}(\mathcal{H} \otimes A)$ takes $a \in A$ to $1 \otimes a$, which is not in $\mathcal{K}(\mathcal{H}) \otimes A = \mathcal{K}(\mathcal{H} \otimes A)$. Thus $(k \otimes \text{id}, 1 \otimes \text{id})$ is automatically Cuntz-Pimsner covariant.

Case 2. $\dim \mathcal{H} = n < \infty$. Then $\mathcal{K}(\mathcal{H}) = B(\mathcal{H})$, so $\varphi = 1 \otimes \text{id}: A \rightarrow \mathcal{L}(\mathcal{H} \otimes A)$ maps injectively into $\mathcal{K}(\mathcal{H}) \otimes A = \mathcal{K}(\mathcal{H} \otimes A)$. Thus $J_{\mathcal{H} \otimes A} = A$ in this case. To check Cuntz-Pimsner covariance, first choose an orthonormal basis $\{u_1, \dots, u_n\}$ for \mathcal{H} , and let $a \in A$. Note that

$$\sum_{i=1}^n \theta_{u_i, u_i} = 1 \quad \text{in } \mathcal{L}(\mathcal{H}).$$

Factor $a = bc^*$ with $b, c \in A$. Then in $\mathcal{L}(A)$ we have

$$\theta_{b,c}d = b\langle c, d \rangle_A = bc^*d = ad,$$

so $\theta_{b,c} = a$. Thus, in $\mathcal{K}(\mathcal{H} \otimes A)$ we have

$$\begin{aligned} \sum_{i=1}^n \theta_{u_i \otimes b, u_i \otimes c} &= \sum_{i=1}^n (\theta_{u_i, u_i} \otimes \theta_{b,c}) \\ &= \sum_{i=1}^n \theta_{u_i, u_i} \otimes a \\ &= 1 \otimes a \end{aligned}$$

$$= \varphi(a),$$

and so

$$\begin{aligned}
(k \otimes \text{id})^{(1)} \circ \varphi(a) &= (k \otimes \text{id})^{(1)} \left(\sum_1^n \theta_{u_i \otimes b, u_i \otimes c} \right) \\
&= \sum_1^n (k \otimes \text{id})^{(1)} (\theta_{u_i \otimes b, u_i \otimes c}) \\
&= \sum_1^n (k \otimes \text{id})(u_i \otimes b) (k \otimes \text{id})(u_i \otimes c)^* \\
&= \sum_1^n (k(u_i) k(u_i)^* \otimes bc^*) \\
&= \sum_1^n k^{(1)}(\theta_{u_i, u_i}) \otimes a \\
&= k^{(1)} \left(\sum_1^n \theta_{u_i, u_i} \right) \otimes a \\
&= k^{(1)}(1) \otimes a \\
&= 1 \otimes a \\
&= (1 \otimes \text{id})(a),
\end{aligned}$$

proving Cuntz-Pimsner covariance in this case.

Thus, in all cases we have a Cuntz-Pimsner covariant representation $(k \otimes \text{id}, 1 \otimes \text{id})$ of $\mathcal{H} \otimes A$ in $\mathcal{O}_{\mathcal{H}} \otimes A$, and therefore we have a homomorphism

$$\Pi := (k \otimes \text{id}) \times (1 \otimes \text{id}): \mathcal{O}_{\mathcal{H} \otimes A} \rightarrow \mathcal{O}_{\mathcal{H}} \otimes A.$$

Since the image of Π contains the generators $\xi \otimes a$ for $\xi \in \mathcal{H}$ and $a \in A$, Π is surjective.

We aim to apply the Gauge-Invariant Uniqueness Theorem [Kat04, Theorem 6.4] to show that Π is injective. Let γ be the gauge action of \mathbb{T} on $\mathcal{O}_{\mathcal{H}}$. Then $\gamma \otimes \text{id}$ is an action of \mathbb{T} on $\mathcal{O}_{\mathcal{H}} \otimes A$, and routine computations show that the pair $(k \otimes \text{id}, 1 \otimes \text{id})$ is compatible with this action. Therefore Π is injective, and hence gives an isomorphism $\mathcal{O}_{\mathcal{H} \otimes A} \simeq \mathcal{O}_{\mathcal{H}} \otimes A$. \square

Remark 4.4. We formulated Corollary 4.2 to get the conclusion regarding \mathcal{O}_X , but since X is isomorphic to the external tensor product of $\ell^2(G/H)$ and A we could deduce other facts as well. For example,

$$\mathcal{K}(X) \simeq \mathcal{K}(\ell^2(G/H) \otimes A)$$

$$\simeq \mathcal{K}(\ell^2(G/H)) \otimes A.$$

Since $\mathcal{K}(X) \simeq C_0(G/H) \rtimes G$ by Rieffel's imprimitivity theorem, we have a tensor-product decomposition of the crossed product:

$$C_0(G/H) \rtimes G \simeq \mathcal{K}(\ell^2(G/H)) \otimes C^*(H).$$

Of course, this is observation not new; for example, since G acts trivially on the open central subgroup H , we could deduce this decomposition from [Gre80, Corollary 2.12].

Remark 4.5. There is a unique continuous action $\alpha: H \rightarrow \text{Aut } \mathcal{O}_{\ell^2(G/H)}$ such that

$$(4.1) \quad \alpha_h(S_{tH}) = S_{htH} \quad \text{for } h \in H, tH \in G/H.$$

This is routine: H acts continuously on the discrete space G/H , giving a strongly continuous unitary representation of H on the Hilbert space $\ell^2(G/H)$, which by universal properties determines a continuous action of H by automorphisms on $\mathcal{O}_{\ell^2(G/H)}$.

5. G DISCRETE

Suppose that G is discrete and H is any subgroup. We identify a group element $s \in G$ with the characteristic function of $\{s\}$, so that

$$c_c(G) = \text{span } G$$

is a dense subspace of the $C^*(H)$ -correspondence X . Similarly, we have $c_c(H) = \text{span } H$, which is a dense *-subalgebra of $A = C^*(H)$.

In the discrete case we will modify our notation for Toeplitz representations of the $C^*(H)$ -correspondence X : we use U rather than π for a representation of A , to remind us that it is the integrated form of a unitary representation of the discrete group H (and we blur the distinction between the unitary representation and its integrated form).

Choose a cross section $\eta: G/H \rightarrow G$, and define $\kappa: H \times G/H \rightarrow H$ by

$$\kappa(h, tH) = \eta(htH)^{-1}h\eta(tH)$$

Lemma 5.1. *With the above notation, κ is a cocycle for the action of H on G/H .*

Proof. This is just the canonical cocycle $G \times G/H \rightarrow H$ restricted to $H \times G/H$. \square

For $s, t \in G$ we have

$$\langle s, t \rangle_A = \begin{cases} s^{-1}t & \text{if } sH = tH \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $h \in H$,

$$\theta_{s,s}sh = s\langle s, sh \rangle_A = sh,$$

while $\theta_{s,s} = 0$ on $\text{span}\{t : t \notin sH\}$.

In the correspondence X , the set of representatives $\{\eta(tH) : tH \in G/H\}$ is orthonormal, and we have

$$(5.1) \quad h\eta(tH) = \eta(htH)\kappa(h, tH) \quad \text{for } h \in H, tH \in G/H.$$

Our analysis of \mathcal{O}_X will depend up whether the index $[G : H]$ is finite or infinite. If $[G : H] < \infty$, then X is algebraically finitely generated, so X is finitely generated projective as a Hilbert A -module, which simplifies things a great deal. Rather than appeal to general theory, though, we show how this works in our special situation. Because G/H is finite, in $\mathcal{L}(X)$ we have

$$\sum_{tH \in G/H} \theta_{\eta(tH), \eta(tH)} = 1.$$

In particular, $\mathcal{K}(X) = \mathcal{L}(X)$. Thus the correspondence X is regular, i.e., $J_X = A$ — of course we already knew this because H is open and has finite index in G . Also, \mathcal{O}_X is unital, and for every Cuntz-Pimsner covariant representation (ψ, U) of X the associated homomorphism $k_X^{(1)}$ of $\mathcal{K}(X)$ is unital.

Proposition 5.2. *Let H be a subgroup of a discrete group G , and let B be a unital C^* -algebra. Then the Cuntz-Pimsner covariant representations of the $C^*(H)$ -correspondence X in B are in 1-1 correspondence with pairs (Ψ, U) , where $\Psi : \mathcal{O}_{\ell^2(G/H)} \rightarrow B$ is a unital homomorphism, $U : H \rightarrow B$ is a unitary homomorphism, and*

$$(5.2) \quad U_h \Psi(S_{tH}) = \Psi(S_{htH}) U_{\kappa(h, tH)} \quad \text{for } h \in H, tH \in G/H.$$

Proof. First suppose that (ψ, U) is a Cuntz-Pimsner covariant representation of X in B . Then for $tH, uH \in G/H$ we have

$$\psi(\eta(tH))^* \psi(\eta(uH)) = U(\langle \eta(tH), \eta(uH) \rangle_A) = \begin{cases} 1 & \text{if } tH = uH \\ 0 & \text{otherwise,} \end{cases}$$

since the set $\{\eta(tH) : tH \in G/H\}$ is orthonormal in the Hilbert A -module X . Thus the $\psi(\eta(tH))$ are isometries with mutually orthogonal ranges.

If $[G : H] < \infty$ then, since the correspondence X is regular and nondegenerate, the homomorphism $\psi^{(1)} : \mathcal{K}(X) \rightarrow B$ is unital, so

$$\sum_{tH \in G/H} \psi(\eta(tH)) \psi(\eta(tH))^* = \sum_{tH \in G/H} \psi^{(1)}(\theta_{\eta(tH), \eta(tH)})$$

$$\begin{aligned}
&= \psi^{(1)}(1) \\
&= 1.
\end{aligned}$$

Thus in all cases there is a unique unital homomorphism $\Psi: \mathcal{O}_{\ell^2(G/H)} \rightarrow B$ such that

$$\Psi(S_{tH}) = \psi(\eta(tH)) \quad \text{for } tH \in G/H.$$

For (5.2), if $h \in H$ and $tH \in G/H$ then by (5.1)

$$\begin{aligned}
U_h \Psi(S_{tH}) &= \psi(h\eta(tH)) = \psi(\eta(htH)\kappa(h, tH)) \\
&= \Psi(S_{htH})U_{\kappa(h, tH)}.
\end{aligned}$$

Now suppose that (Ψ, U) is a pair as in the Proposition. Since the map $(tH, h) \mapsto \eta(tH)h$ from $G/H \times H$ to G is bijective, the set

$$\{\eta(tH)h : tH \in G/H, h \in H\}$$

is a linear basis for $c_c(G)$, so there is a unique linear map $\psi: c_c(G) \rightarrow B$ such that

$$\psi(\eta(tH)h) = \Psi(S_{tH})U_h.$$

Since X is the completion of the $c_c(H)$ -precorrespondence $c_c(G)$, the following computations imply that the pair (ψ, U) is a Toeplitz representation of X in B : for $tH, uH \in G/H$ and $h, k \in H$,

$$\begin{aligned}
\psi(\eta(tH)h)^* \psi(\eta(uH)k) &= (\Psi(S_{tH})U_h)^* \Psi(S_{uH})U_k \\
&= U_h^* \Psi(S_{tH})^* \Psi(S_{uH})U_k
\end{aligned}$$

which, since the $\Psi(S_{tH})$ are isometries with mutually orthogonal ranges, is 0 if $tH \neq uH$, and if $tH = uH$ it is

$$\begin{aligned}
&= U_h^* U_k \\
&= U_{h^{-1}k} \\
&= U(\langle \eta(tH)h, \eta(tH)k \rangle_A),
\end{aligned}$$

and

$$\begin{aligned}
U_h \psi(\eta(tH)k) &= U_h \Psi(S_{tH})U_k \\
&= \Psi(S_{htH})U_{\kappa(h, tH)}U_k \\
&= \Psi(S_{htH})U_{\kappa(h, tH)k} \\
&= \psi(\eta(htH)\kappa(h, tH)k) \\
&= \psi(h\eta(tH)k).
\end{aligned}$$

If $[G : H] = \infty$, the Toeplitz representation (ψ, U) is automatically Cuntz-Pimsner covariant. On the other hand, if $[G : H] < \infty$, we must verify Cuntz-Pimsner covariance: for $h \in H$, since

$$\begin{aligned}\varphi(h) &= \varphi(h)1 \\ &= \sum_{tH \in G/H} \varphi(h)\theta_{\eta(tH), \eta(tH)} \\ &= \sum_{tH \in G/H} \theta_{h\eta(tH), \eta(tH)},\end{aligned}$$

we have

$$\begin{aligned}\psi^{(1)}(\varphi(h)) &= \sum_{tH \in G/H} \psi^{(1)}(\theta_{h\eta(tH), \eta(tH)}) \\ &= \sum_{tH \in G/H} \psi(h\eta(tH))\psi(\eta(tH))^* \\ &= \sum_{tH \in G/H} U_h \Psi(S_{tH}) \Psi(S_{tH})^* \\ &= U_h.\end{aligned}$$

Thus we have defined procedures going both ways: starting with a Cuntz-Pimsner covariant homomorphism (ψ, U) of X in B , we produced a pair (Ψ, U) as in the Proposition, and on the other hand, starting with a pair (Ψ, U) as in the Proposition, we produced a Cuntz-Pimsner covariant homomorphism (ψ, U) of X in B . We verify that these procedures are inverse to each other: first, if we use (ψ, U) to produce (Ψ, U) , and then in turn use that to produce (ψ', U) , then for all $tH \in G/H, h \in H$ we have

$$\begin{aligned}\psi'(\eta(tH)h) &= \Psi(S_{tH})U_h \\ &= \psi(\eta(tH))U_h \\ &= \psi(\eta(tH)h),\end{aligned}$$

and it follows that $\psi' = \psi$. On the other hand, if we use (Ψ, U) to produce (ψ, U) , and then in turn use that to produce (Ψ', U) , then for all $tH \in G/H$ we have

$$\Psi'(S_{tH}) = \psi(\eta(tH)) = \Psi(S_{tH}),$$

and it follows that $\Psi' = \Psi$. \square

Remark 5.3. If $[G : H] < \infty$, then the correspondence X is nondegenerate and regular, so Corollary 2.5 applies, and hence the Cuntz-Pimsner covariant representations of X on a Hilbert space \mathcal{H} are in

1-1 correspondence with the pairs (U, V) , where U is a unitary representation of H on \mathcal{H} and $V: X \otimes_A \mathcal{H} \rightarrow \mathcal{H}$ implements a unitary equivalence between $X\text{-Ind } U$ and U . Comparing with Proposition 5.2 above, it makes sense to ask, given a pair (Ψ, U) , where Ψ is a unital representation of $\mathcal{O}_{\ell^2(G/H)}$ on \mathcal{H} and U is a unitary representation of H on \mathcal{H} satisfying (5.2), what is the associated unitary intertwiner V ? Comparing the proofs of Corollary 2.5 and Proposition 5.2, it is easy to see that $V: X \otimes_A \mathcal{H} \rightarrow \mathcal{H}$ is the unique bounded linear map such that

$$V(\eta(tH) \otimes \xi) = \Psi(S_{tH})\xi \quad \text{for } tH \in G/H, \xi \in \mathcal{H}.$$

However, it turns out that it would not save any time or effort to use Corollary 2.5 to help prove Proposition 5.2.

Remark 5.4. If $[G : H] < \infty$, then Proposition 5.2 is closely related to (indeed, essentially a special case of) [KPW98, Discussion on page 298]. To see this, recall from [KPW98] that a finite set $\{y_1, \dots, y_n\} \subset X$ is called a *basis* for X if $x = \sum_{i=1}^n y_i \langle y_i, x \rangle_A$ for all $x \in X$, and then for all $a \in A$ and all j we have

$$\varphi(a)y_j = \sum_{i=1}^n y_i a_{ij},$$

where $a_{ij} = \langle y_i, \varphi(a)y_j \rangle_A$. [KPW98] then shows that \mathcal{O}_X is the universal C^* -algebra generated by A and n elements S_1, \dots, S_n satisfying

- $S_i^* S_j = \langle y_i, y_j \rangle_A$,
- $\sum_{i=1}^n S_i S_i^* = 1$, and
- $a S_j = \sum_{i=1}^n S_i a_{ij}$ for all $a \in A$ and $j = 1, \dots, n$.

In our setting, we have $A = C^*(H)$, and we are assuming that H has finite index n in G . For all $tH \in G/H$ and $h \in H$ we have

$$\begin{aligned} \sum_{uH \in G/H} \eta(uH) \langle \eta(uH), \eta(tH)h \rangle_A &= \eta(tH) \langle \eta(tH), \eta(tH) \rangle_A h \\ &= \eta(tH)h, \end{aligned}$$

so $\{\eta(uH)\}_{uH \in G/H}$ is a basis of the $C^*(H)$ -correspondence X . This basis is orthonormal, since

$$\langle \eta(uH), \eta(tH) \rangle_A = \begin{cases} 1 & \text{if } uH = tH \\ 0 & \text{if } uH \neq tH. \end{cases}$$

Thus by [KPW98] \mathcal{O}_X is universally generated by A and a Cuntz family of isometries $\{S_{uH}\}_{uH \in G/H}$ satisfying

$$h S_{tH} = \sum_{uH \in G/H} S_{uH} a_{uH, tH},$$

where

$$a_{uH,tH} = \langle \eta(uH), h\eta(tH) \rangle_A.$$

Now,

$$h\eta(tH) = \eta(htH)\kappa(h, tH),$$

so

$$\begin{aligned} \langle \eta(uH), h\eta(tH) \rangle_A &= \langle \eta(uH), \eta(htH)\kappa(h, tH) \rangle_A \\ &= \begin{cases} \kappa(h, tH) & \text{if } uH = htH \\ 0 & \text{if } uH \neq htH, \end{cases} \end{aligned}$$

and so the scheme of [KPW98] says that

$$hS_{tH} = S_{htH}\kappa(h, tH),$$

which is the condition (5.2) of Proposition 5.2.

Remark 5.5. If the cocycle $\kappa: H \times G/H \rightarrow H$ is given by $\kappa(h, tH) = h$, then the Cuntz-Pimsner algebra \mathcal{O}_X is isomorphic to the crossed product $\mathcal{O}_{G/H} \rtimes_{\alpha} H$, where $\alpha: H \rightarrow \text{Aut } \mathcal{O}_{G/H}$ is the action defined by (4.1). This happens if and only if the cross section $\eta: G/H \rightarrow G$ is equivariant for the left H -actions:

$$h\eta(tH) = \eta(htH) \quad \text{for } h \in H, tH \in G/H.$$

But this can never happen (unless $H = \{1\}$), since H acts freely on G but has a fixed point in G/H .

Corollary 5.6. *Let G be discrete, let E be the directed graph with one vertex and edge set $E^1 = G/H$, and let H act on E by fixing the vertex and acting on the edges by left translation on the homogeneous space. Then κ is a cocycle for the action of H on the graph E in the sense of [BKQ, Definition 3.3], and the correspondence X is isomorphic to the associated correspondence Y^{κ} of [BKQ, Definition 3.6], and so the Cuntz-Pimsner algebra \mathcal{O}_X is isomorphic to the Exel-Pardo algebra $\mathcal{O}_{Y^{\kappa}}$ of [EP, Definition 3.8]. If H has finite index in G , then the graph E is finite, and so X is isomorphic to the associated correspondence M of [EP, Section 10], and so \mathcal{O}_X is isomorphic to the algebra $\mathcal{O}_{H,G/H}$ of [EP, Definition 3.2].*

Proof. Recall from [BKQ, Definition 3.6] that the correspondence Y^{κ} is constructed as follows: first of all, since E has only one vertex we can identify $c_0(E^0) \rtimes H$ with $A = C^*(H)$. Now give the set $G/H \times H$ the following operations, for $tH, uH \in G/H, h, k \in H$:

- $(tH, k)h = (tH, kh);$

- $\langle (tH, h), (uH, k) \rangle_A = \begin{cases} h^{-1}k & \text{if } tH = uH \\ 0 & \text{otherwise;} \end{cases}$
- $h(tH, k) = (htH, \kappa(h, tH)k).$

Then the linear span $c_c(G/H \times H)$ becomes a $c_c(H)$ -precorrespondence, whose completion is Y^κ . It is routine to check that the map

$$(tH, h) \mapsto \eta(tH)h: G/H \times H \rightarrow G$$

integrates to an isomorphism $Y^\kappa \simeq X$ as $C^*(H)$ -correspondences. \square

Remark 5.7. Since the graph E described in Corollary 5.6 has only one vertex, we are actually in the situation of a self-similar group action, so \mathcal{O}_X is isomorphic to the C^* -algebra $\mathcal{O}_{(H, G/H)}$ of [Nek09, Definition 3.1] (see also [LRRW14, Proposition 3.2 and Remark 3.6]).

Remark 5.8. The Cuntz-Pimsner algebra \mathcal{O}_X does not have anything directly to do with the cross section η , but obviously the Exel-Pardo correspondence Y^κ does. So Corollary 5.6 raises an obvious issue: how is the independence of \mathcal{O}_X upon η reflected in \mathcal{O}_{Y^κ} ? More precisely, if we choose another cross section $\eta': G/H \rightarrow G$, and use it to define another cocycle $\kappa': H \times G/H \rightarrow H$, then clearly the Exel-Pardo algebras \mathcal{O}_{Y^κ} and $\mathcal{O}_{Y^{\kappa'}}$ must be isomorphic, since they are both isomorphic to \mathcal{O}_X ; could we have predicted this just using the theory of cocycles? The answer is yes, because the cocycles κ and κ' will be cohomologous. For completeness, we include a reminder: Two cocycles κ, κ' for the action of H on G/H are called *cohomologous* if there is a map $\nu: G/H \rightarrow H$ such that

$$\kappa'(h, tH) = \nu(htH)^{-1} \kappa(h, tH) \nu(tH) \quad \text{for } h \in H, tH \in G/H.$$

Let κ be defined using the cross section $\eta: G/H \rightarrow G$ as above. Given a map $\nu: G/H \rightarrow H$, we get another cross section

$$\eta'(tH) = \eta(tH) \nu(tH),$$

and conversely, given another cross section $\eta': G/H \rightarrow G$, we get a map $\nu: G/H \rightarrow H$ defined by

$$\nu(tH) = \eta(tH)^{-1} \eta'(tH),$$

and it is well-known that the two cocycles associated to the cross sections η, η' are cohomologous:

$$\begin{aligned} \kappa'(h, tH) &= \eta'(htH)^{-1} h \eta'(tH) \\ &= (\eta(htH) \nu(htH))^{-1} h \eta(tH) \nu(tH) \\ &= \nu(htH)^{-1} \eta(htH)^{-1} h \eta(tH) \nu(tH) \end{aligned}$$

$$= \nu(htH)^{-1}\kappa(h, tH)\nu(tH).$$

It then follows that the two correspondences Y^κ and $Y^{\kappa'}$, and hence the associated Exel-Pardo algebras \mathcal{O}_{Y^κ} and $\mathcal{O}_{Y^{\kappa'}}$, are isomorphic [BKQ, Theorem 4.8].

It might be of interest to interpret the above in terms of a classification result of Zimmer: The orbits of the action of H on G/H are the double cosets in $H \backslash G / H$. Thus the cocycle κ is uniquely determined by the restricted cocycles $\kappa|_{H \times H_{tH}}$. For each coset $tH \in G/H$ the stability subgroup of the action of H is

$$H_{tH} := H \cap \eta(tH)H\eta(tH)^{-1} = \{h \in H : htH = tH\}.$$

Then the action on the orbit HtH is conjugate to the action of H on the coset space H/H_{tH} , and a result of Zimmer [Zim84, 4.2.13] (also recorded in a form more convenient for our purposes in [BKQ, Lemma 2.8]) classifies those: the cohomology classes of such cocycles are in 1-1 correspondence with the set of conjugacy classes of homomorphisms from H_{tH} to H . The restricted cocycle

$$\kappa_{tH}: H \times H/H_{tH} \rightarrow H$$

is given by

$$\kappa_{tH}(h, kH_{tH}) = \kappa(h, ktH) \quad \text{for } h, k \in H.$$

The homomorphism $\tau_{tH}: H_{tH} \rightarrow H$ associated with the restricted cocycle κ_{tH} is given by

$$\tau_{tH}(h) = \kappa_{tH}(h, H_{tH}) = \kappa(h, tH) \quad \text{for } h \in H_{tH}.$$

Conversely, starting with a homomorphism $\tau: H_{tH} \rightarrow H$, the associated cocycle $\mu: H \times H/H_{tH} \rightarrow H$ is constructed by first choosing a cross section $\gamma: H/H_{tH} \rightarrow H$ with $\gamma(H_{tH}) = 1$, and then defining

$$\mu(h, kH_{tH}) = \gamma(hkH_{tH})^{-1}h\gamma(kH_{tH}).$$

In the case of the Rieffel A -correspondence X , the unique cohomology class of cocycles is determined by the inclusion homomorphisms $H_{tH} \hookrightarrow H$ for each $tH \in G/H$.

Question 5.9. Corollary 5.6 leads to another obvious question: what Exel-Pardo algebras arise in this manner? Put another way, what cocycles κ arise from the above procedure? More precisely, if we start with an action of H on a set T and a cocycle $\kappa: H \times T \rightarrow H$ for this action, when will there exist a group G containing H as a subgroup such that G/H can be identified with T and κ arises as above? There is one obvious obstruction: there must be at least one fixed point in T , since H fixes the coset H in G/H . Are there any other obstructions? For

example, can we realize all of Katsura's algebras $\mathcal{O}_{A,B}$ [Kat08] (also see [EP, Example 3.4]), which include all Kirchberg algebras in the UCT class?

Another obstruction is the cohomology class of the cocycle: as we mentioned in Remark 5.8, for every double coset HtH the cohomology class of the restricted cocycle corresponds to the inclusion homomorphism $H_{tH} \hookrightarrow H$. Thus it would appear that we do not get all cocycles.

6. H COMPACT

Recall from [Dix77, Section 4.1, Addendum 4.7.21] that a C^* -algebra is called *elementary* if it is isomorphic to the algebra of compact operators on a Hilbert space, and *dual* if it is a c_0 -direct sum

$$A = \bigoplus_{\mu \in \Omega} A_\mu$$

of elementary algebras. We can identify the spectrum \widehat{A} of A with the set Ω . Any two dual algebras with spectrum Ω are Morita equivalent, and we need a particular consequence regarding Cuntz-Pimsner algebras.

Let A and B be dual algebras with spectrum Ω , and with component elementary algebras A_μ and B_μ . For each $\mu \in \Omega$ choose an $A_\mu - B_\mu$ imprimitivity bimodule M_μ , and define an $A - B$ imprimitivity bimodule M by

$$M = \bigoplus_{\mu \in \Omega} M_\mu.$$

Let X be a faithful nondegenerate A -correspondence, and define a faithful nondegenerate B -correspondence Y by

$$Y = M^* \otimes_A X \otimes_A M.$$

By [MS00, Theorem 3.5], the Cuntz-Pimsner algebras \mathcal{O}_X and \mathcal{O}_Y are Morita equivalent.

In the particular case where all the B_μ are 1-dimensional, so that B is commutative, by [KPQ12, Theorem 1.1] Y is isomorphic to the correspondence associated to a directed graph E with vertex set Ω and in which for $\mu, \nu \in \Omega$ the cardinality of $\mu E^1 \nu$ is the dimension of the Hilbert space $p_\mu Y p_\nu$, where p_μ denotes the identity element of B_μ , regarded as a central projection in B . Thus $\mathcal{O}_Y \simeq C^*(E)$, and hence \mathcal{O}_X is Morita equivalent to the graph algebra $C^*(E)$. For this to be useful, we would like to be able to find the edges of the graph E directly using the A -correspondence X . For each $\mu \in \Omega$ choose associated irreducible representations π_μ of A and τ_μ of B . Then by the

construction of E in [KPQ12], the cardinality of $\mu E^1 \nu$ coincides with the multiplicity of τ_μ in the induced representation $Y\text{-Ind } \tau_\nu$. Thus we expect the following:

Lemma 6.1. *For all $\mu, \nu \in \Omega$, the cardinality of $\mu E^1 \nu$ equals the multiplicity of π_μ in the representation $X\text{-Ind } \pi_\nu$.*

Proof. It suffices to show that for all $\mu, \nu \in \Omega$ the multiplicity of π_μ in $X\text{-Ind } \pi_\nu$ equals the multiplicity of τ_μ in $Y\text{-Ind } \tau_\nu$. This is almost obvious, and we include the routine computation. We have unitary equivalences

$$M\text{-Ind } \tau_\mu \simeq \pi_\mu \quad \text{for all } \mu \in \Omega.$$

Fix $\nu \in \Omega$, and decompose $X\text{-Ind } \pi_\nu$ into irreducibles:

$$X\text{-Ind } \pi_\nu \simeq \bigoplus_{\mu \in \Omega} n_\mu \pi_\mu,$$

where n_μ is the multiplicity of π_μ in $X\text{-Ind } \pi_\nu$. Then we have

$$\begin{aligned} Y\text{-Ind } \tau_\nu &\simeq M^*\text{-Ind } X\text{-Ind } M\text{-Ind } \tau_\nu \\ &\simeq M^*\text{-Ind } X\text{-Ind } \pi_\nu \\ &\simeq M^*\text{-Ind } \bigoplus_{\mu \in \Omega} n_\mu \pi_\mu \\ &\simeq \bigoplus_{\mu \in \Omega} M^*\text{-Ind } n_\mu \pi_\mu \\ &\simeq \bigoplus_{\mu \in \Omega} n_\mu M^*\text{-Ind } \pi_\mu \\ &\simeq \bigoplus_{\mu \in \Omega} n_\mu \tau_\mu, \end{aligned}$$

and the result follows. \square

Corollary 6.2. *When X is a faithful nondegenerate correspondence over a dual C^* -algebra A , the Cuntz-Pimsner algebra \mathcal{O}_X is Morita equivalent to the graph algebra $C^*(E)$ of a directed graph with vertex set \hat{A} and in which, for all $\pi, \sigma \in \hat{A}$, the number of edges from σ to π is the multiplicity of π in $X\text{-Ind } \sigma$. If A is commutative then $\mathcal{O}_X \simeq C^*(E)$.*

Now let H be a compact subgroup of G , let $A = C^*(H)$, and let X be the A -correspondence for Rieffel induction. Then A is a dual algebra, so by the above we have:

Corollary 6.3. *When H is compact the Cuntz-Pimsner algebra \mathcal{O}_X is Morita equivalent to the graph algebra $C^*(E)$ of a directed graph with vertex set \widehat{H} and in which, for all $U, V \in \widehat{H}$, the number of edges from V to U is the multiplicity of U in $X\text{-Ind } V$. If H is abelian then $\mathcal{O}_X \simeq C^*(E)$.*

Question 6.4. Which directed graphs arise as in Corollary 6.3? It follows from [Fel64, Corollary 3 of Theorem 5.5] that any such graph has at least one loop edge at every vertex.

Remark 6.5. One could push the above machinery further, to classify up to isomorphism all faithful nondegenerate A -correspondences, but since we do not need this for our results we only give a very rough outline. As above, let $B = \bigoplus_{\mu \in \Omega} B_\mu$ be a commutative C^* -algebra with spectrum Ω . For each $\mu \in \Omega$ there is up to isomorphism a unique $A_\mu - B_\mu$ imprimitivity bimodule M_μ , namely any Hilbert space of the appropriate dimension, and as before let $M = \bigoplus_{\mu \in \Omega} M_\mu$ be the associated $A - B$ imprimitivity bimodule. Every faithful nondegenerate A -correspondence X gives rise to a faithful nondegenerate B -correspondence $Y = M^* \otimes_A X \otimes_A M$, and this process is reversible:

$$M \otimes_B M^* \otimes_A X \otimes_A M \otimes_B M^* \simeq A \otimes_A X \otimes_A A \simeq X,$$

since $AX = X$. The B -correspondence Y is characterized up to isomorphism by the directed graph E with vertex set Ω and the number of edges from ν to μ given by the dimension of the Hilbert space $p_\mu Y p_\nu$, where p_μ is the identity element of B_μ , regarded as a minimal projection in B . Up to isomorphism, the A -correspondence X can be decomposed as

$$\bigoplus_{\mu, \nu \in \Omega} M_\mu^* \otimes_{B_\mu} p_\mu Y p_\nu \otimes_{B_\nu} M_\nu^*.$$

7. EXAMPLES

Interesting examples arise already with finite groups. So, let H be a subgroup of a finite group G . Since H is finite, it is compact, so by Corollary 6.3 the Cuntz-Pimsner algebra \mathcal{O}_X is Morita equivalent to the C^* -algebra of a directed graph E with $E^0 = \widehat{H}$ and in which, for $U, V \in \widehat{H}$, the cardinality of UE^1V is the multiplicity of U in $X\text{-Ind } V$. To compute these multiplicities, we appeal to Mackey's Subgroup Theorem [Mac52, Theorem 7.1], which in our situation can be expressed in the form

$$X\text{-Ind } V \simeq \bigoplus_{HsH \in H \backslash G / H} \text{Ind}_{H_s}^H V^s,$$

where

$$H_s = H \cap s^{-1}Hs \quad \text{and} \quad V^s = V \circ \text{Ad } s|_{H_s},$$

and where in the direct sum we take one representative s from each double coset HsH .

As we observed in Section 3, the cases $H = \{1\}$ or $H = G$ are boring, so we focus on proper nontrivial subgroups. The case $H = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is already interesting, so we examine it in some detail. First note that, since \mathbb{Z}_2 is abelian, by Corollary 6.3 we actually have $\mathcal{O}_X \simeq C^*(E)$ for the above directed graph E .

If the subgroup $H = \mathbb{Z}_2$ is normal, then it is central, so by Corollary 4.2 we have $\mathcal{O}_X \simeq \mathcal{O}_{[G:H]} \otimes \mathbb{C}^2$. So we assume from now on that H is nonnormal. Then the action of H on G/H has at least one fixed point (namely H) and at least one 2-element orbit. Let

r be the number of fixed points in G/H , and
 q the number of 2-element orbits.

Note that r is the index $[Z_G(H) : H]$ of H in its centralizer $Z_G(H)$, and $[G : H] = r + 2q$.

What pairs (r, q) can occur?

Proposition 7.1. *With the above notation, a pair (r, q) of positive integers can arise if and only if $r \mid 2q$.*

Proof. First suppose that H is a proper nonnormal subgroup of a finite group G with $H \simeq \mathbb{Z}_2$. As above, put $r = [Z_G(H) : H]$, and let $[G : H] = r + 2q$, so that q is a positive integer. We have $|G| = 2r + 4q$. Also,

$$|Z_G(H)| = 2r,$$

which must divide $|G|$, i.e., $2r \mid (2r + 4q)$. Thus $2r \mid 4q$, so $r \mid 2q$.

Conversely, let r and q be positive integers with $r \mid 2q$, say $2q = mr$. We must show that there exists a finite group G containing a subgroup $H \simeq \mathbb{Z}_2$ such that

$$[Z_G(H) : H] = r \quad \text{and} \quad [G : H] = r + 2q.$$

Case 1. m is even. Put

$$G = \mathbb{Z}_r \times (\mathbb{Z}_{m+1} \rtimes \mathbb{Z}_2),$$

where $H = \mathbb{Z}_2$ acts on \mathbb{Z}_{m+1} by the automorphism $n \mapsto -n$. Since $m + 1$ is odd, this automorphism has no fixed points other than the identity element 0, so

$$Z_G(H) = \mathbb{Z}_r \times \mathbb{Z}_2,$$

and hence $[Z_G(H) : H] = r$. Further,

$$[G : H] = r(m+1) = r + rm = r + 2q.$$

Case 2. m is odd. Then r is even, say $r = 2j$. Put

$$G = \mathbb{Z}_j \times (\mathbb{Z}_{2(m+1)} \rtimes \mathbb{Z}_2),$$

where again $H = \mathbb{Z}_2$ acts on $\mathbb{Z}_{2(m+1)}$ by $n \mapsto -n$. In this case, the fixed-point subgroup under this action is $\{0, m+1\}$, so

$$Z_G(H) = \mathbb{Z}_j \times \{0, m+1\} \times \mathbb{Z}_2,$$

and hence $[Z_G(H) : H] = j \cdot 2 = r$. Further,

$$[G : H] = j \cdot 2(m+1) = r(m+1) = r + 2q,$$

as desired. \square

Now we continue the investigation of the directed graph E , begun in the first paragraph of this section. Note that for each $s \in Z_G(H)$ we have $H_s = H$ and $V^s = V$, so the fixed points in G/H contribute a summand rV in $X\text{-Ind } V$.

Each 2-element orbit in G/H is a disjoint union $sH \sqcup hsH$, where $s \notin Z_G(H)$ and h is the generator of H . We have $H_s = \{1\}$ and consequently V^s is (equivalent to) the trivial character 1, and so

$$\text{Ind}_{H_s}^H V^s = \text{Ind}_{\{1\}}^H 1,$$

which is the regular representation λ_H of H . As H is a finite abelian group, we have

$$\lambda_H \simeq \bigoplus_{U \in \widehat{H}} U.$$

Combining, we see that for each $V \in \widehat{H}$,

$$X\text{-Ind } V \simeq rV \oplus q \bigoplus_{U \in \widehat{H}} U = (r+q)V \oplus qU,$$

where U is the character of H different from V . Consequently, the associated graph E has the form

$$\begin{array}{c} (r+q) \curvearrowright U \begin{array}{c} \xleftarrow{(q)} \\ \xrightarrow{(q)} \end{array} V \curvearrowright (r+q) \end{array}$$

where U, V are the two characters of H , and where a number in parentheses indicates the number of edges from the first vertex to the second. Because we are assuming that H is a proper non-normal subgroup, we have $q > 0$. Thus the graph E is finite and transitive (meaning that $vE^1w \neq \emptyset$ for all $v, w \in E^0$), and every cycle has an entry, so

by [KPR98, Corollary 3.11] $C^*(E)$ is unital, simple, and purely infinite. By [Rae05, Remark 4.3], $C^*(E)$ is nuclear and in the bootstrap class. Thus \mathcal{O}_X , being isomorphic to $C^*(E)$, is classifiable up to Morita equivalence by its K -theory, according to the classification theorem of Kirchberg and Phillips [KP00, Phi00]. In fact, since \mathcal{O}_X is unital, it is classifiable up to isomorphism by K_0 , K_1 , and the class $[1]_0$ in K_0 of the identity $1_{\mathcal{O}_X}$.

Remark 7.2. For any prime ℓ , we can consider the case $H = \mathbb{Z}_\ell$, and we get a graph similar to the above, but with ℓ vertices; it will still have q edges between any two distinct vertices and $r + q$ loop edges at each vertex.

To compute the K -theory, by [Rae05, Theorem 7.16] we can use the vertex matrix A , indexed by E^0 , where the ij -entry is the number of edges from the j th vertex to the i th one. And then the algorithm tells us that, identifying the matrix $B := A^t - 1$ with an endomorphism of the free abelian group \mathbb{Z}^{E^0} , we have

$$\begin{aligned} K_1(C^*(E)) &= \ker B \\ K_0(C^*(E)) &= \operatorname{coker} B = \mathbb{Z}^{E^0} / B\mathbb{Z}^{E^0}. \end{aligned}$$

(The usual formulation involves $1 - A^t$, but in our case the matrix $A^t - 1$ is more convenient, and the results are the same.) In our situation we have $E^0 = \{U, V\}$, and

$$A = \begin{pmatrix} r + q & q \\ q & r + q \end{pmatrix},$$

and so

$$B = A^t - 1 = \begin{pmatrix} r + q - 1 & q \\ q & r + q - 1 \end{pmatrix}.$$

Let $p = r + q - 1$. Then $p \geq q > 0$, and we have

$$\begin{aligned} K_1 &= \ker B \\ K_0 &= \mathbb{Z}^2 / B\mathbb{Z}^2. \end{aligned}$$

Since the graph algebra $C^*(E)$ is unital, we must compute the class $[1]_0$ in K_0 of the identity $1_{C^*(E)}$. For this, we need to compute the classes $[p_v]_0$ of the vertex projections and then add them up. In our case, we have

$$[1]_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathbb{Z}^2.$$

To compute the cokernel of B , we appeal to the standard theory which identifies it with a direct sum of abelian groups via computing

the Smith normal form of B . We recall from eg. [MM64, Section 3.22] how this works. Let $B \in M_n(\mathbb{Z})$, and suppose B has rank k for some $k \leq n$. For each $j = 1, \dots, k$, if B has at least one nonzero j -square subdeterminant, define f_j as the g.c.d. of all j -th order subdeterminants of B . Set $f_0 = 1$. Note that f_{j-1} divides f_j for all $j = 1, \dots, k$. The Smith normal form of B is the matrix N with entries $n_{jj} = q_j$, where

$$q_j := f_j / f_{j-1}$$

for $j = 1, \dots, k$ and zero in all other entries. Note that q_j divides q_{j+1} for all $j = 1, \dots, k$. Moreover, there are invertible matrices $C, D \in M_n(\mathbb{Z})$ such that $B = CND$ and the map $x \rightarrow C^{-1}x$ on \mathbb{Z}^n induces an isomorphism

$$\Phi: \mathbb{Z}^n / B\mathbb{Z}^n \rightarrow \mathbb{Z}^n / N\mathbb{Z}^n.$$

To compute the class of the identity in K_0 we will compute its image in $\mathbb{Z}^n / N\mathbb{Z}^n$ under Φ . To get explicit formulas, we split up the analysis into several cases.

Case 1. $r = 1$. Then $B = \begin{pmatrix} q & q \\ q & q \end{pmatrix}$. Thus $K_1 = \ker B$ is the cyclic subgroup of \mathbb{Z}^2 generated by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so $K_1 \simeq \mathbb{Z}$.

Clearly, B has rank 1. In the above notation we have $B = CND$ for

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Denote by (m, n) the transpose of a column-vector $\begin{pmatrix} m \\ n \end{pmatrix}$ in \mathbb{Z}^2 . The map $(m, n) \mapsto (m \pmod{q}, n)$ has kernel $N\mathbb{Z}^2$, and so induces an isomorphism

$$\Psi: \mathbb{Z}^2 / N\mathbb{Z}^2 \cong \mathbb{Z}_q \oplus \mathbb{Z}.$$

Composing Ψ with Φ gives an isomorphism

$$K_0 \simeq \mathbb{Z}_q \oplus \mathbb{Z},$$

and since C^{-1} carries $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we have, in $\mathbb{Z}_q \oplus \mathbb{Z}$,

$$[1]_0 = (1, 0).$$

By [Spi12, Theorem 4.8 (3)], the C^* -algebra of the category of paths given by the positive submonoid Λ of the Baumslag-Solitar group

$$B(1, q+1) = \langle a, b \mid ab = b^{q+1}a \rangle$$

is UCT Kirchberg and has K -theory $(\mathbb{Z}_q \oplus \mathbb{Z}, \mathbb{Z})$, with $[1]_0 = (1, 0)$, and hence when $r = 1$ we have $\mathcal{O}_X \simeq C^*(\Lambda)$.

Example 7.3. Here is one of the simplest examples of the above: let $H = \mathbb{Z}_2$ as a subgroup of the group $G = S_3$ of permutations of a 3-element set, and let X be the associated $C^*(H)$ -correspondence. It

follows from the above analysis that \mathcal{O}_X is isomorphic to the algebra of the following graph:



With the above notation, we have $r = q = 1$, so the K -groups of \mathcal{O}_X are both \mathbb{Z} . The crossed product of $PSL(2, \mathbb{Z})$ acting on the boundary of the upper half plane, and the Ruelle algebra associated to the 2-adic solenoid are purely infinite simple C^* -algebras with this K -theory [LS96, Application 15], so \mathcal{O}_X is Morita equivalent to either of these.

Case 2. $r > 1$. We have

$$B = \begin{pmatrix} p & q \\ q & p \end{pmatrix},$$

where $p > q > 0$. Then

$$K_1 = \ker B = 0.$$

We turn to computing K_0 . Since B has rank two, we find that $f_0 = 1$, $f_1 = \gcd(p, q)$ and $f_2 = \det B = p^2 - q^2$. Denote $d = p^2 - q^2$.

We first suppose that p and q are coprime, so that $f_1 = 1$. The Euclidean algorithm gives $s, t \in \mathbb{Z}$ such that

$$sp + tq = 1.$$

The Smith normal form of B and the associated invertible matrices C, D are given as follows

$$C = \begin{pmatrix} p & -t \\ q & s \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad D = \begin{pmatrix} 1 & tp + sq \\ 0 & 1 \end{pmatrix}.$$

The map $(m, n) \mapsto (0, n \pmod{d})$ induces an isomorphism

$$\Psi: \mathbb{Z}^2 / N\mathbb{Z}^2 \rightarrow \mathbb{Z}_1 \oplus \mathbb{Z}_d,$$

and the composition $\Psi \circ \Phi$ gives an isomorphism

$$K_0 \simeq \mathbb{Z}_1 \oplus \mathbb{Z}_d = \mathbb{Z}_d.$$

Since the isomorphism Φ carries the class of the identity in K_0 into

$$C^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} s & t \\ -q & p \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} s + t \\ p - q \end{pmatrix},$$

it follows that the image of $[1]_0$ in \mathbb{Z}_d is identified as

$$[1]_0 = p - q.$$

Now,

$$d = p^2 - q^2 = (p - q)(p + q),$$

so $[1]_0$ divides the order of the cyclic group K_0 . If $p - q = 1$, then it is a generator of K_0 , and so

$$\mathcal{O}_X \simeq \mathcal{O}_{d+1}.$$

On the other hand, if $p - q > 1$, then

$$\mathcal{O}_X \simeq M_{p-q} \otimes \mathcal{O}_{d+1}.$$

Now suppose that p and q are not coprime. Rename $a = f_1 = \gcd(p, q)$ and note that the Smith normal form of B is the matrix

$$N = \begin{pmatrix} a & 0 \\ 0 & d/a \end{pmatrix}.$$

Write $p = au$ and $q = av$. Then u and v are coprime, and $d = a^2g$, where $g = u^2 - v^2$ is the determinant of the matrix

$$B_1 = \begin{pmatrix} u & v \\ v & u \end{pmatrix}.$$

We then have $B = aB_1$, where the analysis of the coprime case applies to B_1 . Choosing $z, w \in \mathbb{Z}$ such that $zu + wv = 1$, the matrix that plays the role of C is now

$$C_1 = \begin{pmatrix} u & -w \\ v & z \end{pmatrix}.$$

By the coprime case, we get an isomorphism

$$K_0 \simeq \mathbb{Z}_a \oplus \mathbb{Z}_{ag}.$$

For the class of the identity in K_0 , in $\mathbb{Z}_a \oplus \mathbb{Z}_{ag}$ we have

$$[1]_0 = (z + w, p - q),$$

where we can find suitable z, w using either $zu + wv = 1$ or $zp + wq = a$.

Example 7.4. If $r = 2$, then $p = q + 1$ is coprime to q . We have $p - q = 1$, $d = p + q = 2q + 1$, $K_0 \simeq \mathbb{Z}_{2q+1}$, $[1]_0 = 1$, and

$$\mathcal{O}_X \simeq \mathcal{O}_{2q+2}.$$

Example 7.5. If $q = 1$, then $r = 2$ (since we must have $r \mid 2q$ and we are assuming that $r > 1$), so this is a special case of Example 7.4: we have $p = 2$, giving

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

$d = 3$, $K_0 \simeq \mathbb{Z}_3$, $[1]_0 = 1$, and

$$\mathcal{O}_X \simeq \mathcal{O}_4.$$

Example 7.6. If $r = q = 2$, then again we are in a special case of Example 7.4, and this time $p = 3$, giving

$$B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$

$d = 5$, $K_0 \simeq \mathbb{Z}_5$, $[1]_0 = 1$, and

$$\mathcal{O}_X \simeq \mathcal{O}_6.$$

Example 7.7. If $r = q > 2$, then $p = 2q - 1$, which is coprime to q since

$$-p + 2q = 1.$$

Thus $p - q = q - 1$, $d = (q - 1)(3q - 1)$, $K_0 \simeq \mathbb{Z}_d$, $[1]_0 = q - 1$, and

$$\mathcal{O}_X \simeq M_{q-1} \otimes \mathcal{O}_{d+1}.$$

Example 7.8. As a special case of Example 7.7, if $r = q = 3$, then $p = 5$, $p - q = 2$, $d = 16$, $K_0 \simeq \mathbb{Z}_{16}$, $[1]_0 = 2$, and

$$\mathcal{O}_X \simeq M_2 \otimes \mathcal{O}_{17}.$$

Example 7.9. If $r = 3$ and $q = 6$, then $p = 8$ is not coprime to q . We have $a = \gcd(p, q) = 2$, $p - q = 2$, $u = \frac{p}{a} = 4$, $v = \frac{q}{a} = 3$, $g = u^2 - v^2 = 7$, and $K_0 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_{14}$. Since

$$8 - 6 = 2,$$

we can take $z = 1$ and $w = -1$, so

$$[1]_0 = (z + w, p - q) = (0, 2).$$

Note that if $r = 3$ then, since $r \mid 2q$ we have $3 \mid q$, so in some sense this is the next biggest example after the preceding one, and the smallest one with p and q not coprime.

Example 7.10. If $r = 3$ and $q = 9$, then $p = 11$ is coprime to q . We have $p - q = 2$, $d = 40$, $K_0 \simeq \mathbb{Z}_{40}$, $[1]_0 = 2$, and

$$\mathcal{O}_X \simeq M_2 \otimes \mathcal{O}_{41}.$$

Example 7.11. If $r = 6$ and $q = 9$, then $p = 14$ is coprime to q , and we have $p - q = 5$, $d = 115$, $K_0 \simeq \mathbb{Z}_{115}$, $[1]_0 = 5$, and

$$\mathcal{O}_X \simeq M_5 \otimes \mathcal{O}_{116}.$$

8. CONNECTION WITH [MRS92b]

If the subgroup H is a compact Lie group, then we can choose a faithful finite-dimensional unitary representation ρ . In this situation, [MRS92b, MRS92a] study the Doplicher-Roberts algebra \mathcal{O}_ρ , and show that it is Morita equivalent to a Cuntz-Krieger algebra — equivalently, a graph algebra, although at the time [MRS92b, MRS92a] were written, the technology of graph C^* -algebras had not yet appeared.

The finite-dimensional Hilbert space \mathcal{H} of the representation ρ can be regarded as an $A - \mathbb{C}$ correspondence, where $A = C^*(H)$ as before, but there does not appear to be a natural way to give \mathcal{H} the structure of an A -correspondence. Nevertheless, something interesting happens: the method that [MRS92b, Section 1] use to construct a graph E from ρ is strikingly similar to our construction in Lemma 6.1. In [MRS92b] the construction is as follows: let R be the set of equivalence classes of irreducible representations of H occurring in the various tensor powers $\rho^{\otimes n}$; if H is finite then $R = \widehat{H}$. The graph E has vertex set R , and for each $\pi_1, \pi_2 \in R$ the number of edges in E from π_2 to π_1 is the multiplicity of π_2 in $\pi_1 \otimes \rho$, whereas in our Lemma 6.1 we define $E^0 = \widehat{H}$, and the number of edges from π_2 to π_1 is the multiplicity of π_1 in $X\text{-Ind } \pi_2$. The similarity is uncanny, particularly because the Hilbert space of $X\text{-Ind } \pi_2$ is $X \otimes_A \mathcal{H}_{\pi_2}$.

Moreover, although in [MRS92b] the construction of the Doplicher-Roberts algebra \mathcal{O}_ρ does not explicitly involve an A -correspondence, in the cases where $R = \widehat{H}$ the graph E with $E^0 = R$ gives a correspondence over $c_0(E^0)$, and which is Morita equivalent to A , and hence the method outlined in Remark 6.5 gives an A -correspondence X with \mathcal{O}_X Morita equivalent to $C^*(E)$, and therefore to \mathcal{O}_ρ . That being said, at present this observation remains little more than a curiosity.

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